Department of Mathematics and Computer Science

Comprehensive Examination–Option I

2007 Spring

Algebra

1. List all nonisomorphic abelian groups of order 72, determine the number of nonisomorphic abelian groups of order 2592, and explain your answer.

2. Let R be a ring with multiplicative identity 1_R.
   a. Suppose $f : R \to S$ is a surjective homomorphism of rings. Prove that S contains a multiplicative identity 1_S and $f(1_R) = 1_S$.
   b. Suppose I is an ideal of R containing a unit. Prove that $I = R$.
   c. Find a nonzero ring homomorphism $f : \mathbf{Z}_3 \to \mathbf{Z}_6$ and use it to conclude that the assertion of part a. fails to hold if f is not surjective.

3. Let R be the commutative ring $\mathbf{Z}[i] = \{a + bi \mid a, b \in \mathbf{Z}\}$ and $I = <2 + i>$, the ideal in $\mathbf{Z}[i]$ generated by the element $2 + i$. Prove that R/I is a field.

4. Suppose T is a nonzero nilpotent linear operator on a vector space V over the field F; *i.e.*, $T^m = 0$ for some integer $m \geq 2$. Prove:
   a. 0 is the only eigenvalue of T.
   b. For each nonzero element $\alpha \in F$, $\alpha + T$ is invertible.

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Real Analysis

1. Suppose $A \subset \mathbf{R}$ and $f : A \to \mathbf{R}$. Complete the definition: f is *uniformly continuous* on A if ..., and using your definition prove that if $a > 1/2$, then the function g defined by

$$g(x) = \frac{5x}{2x-1}$$

is uniformly continuous on $[a, \infty)$.

2. Consider the sequence $(x_n)_{n=1}^{\infty}$ defined by setting $x_1 = 1,\ x_2 = 2$, and then, recursively,

$$x_n = \frac{x_{n-1} + x_{n-2}}{2}$$

for each integer $n \geq 3$.
   a. Show that (x_n) is a Cauchy sequence.
   b. Calculate the limit of (x_n).
   Hint: Compare to $1 + 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - + \cdots$.

3. State and prove the intermediate value theorem for functions $f : [a, b] \to \mathbf{R}$ where $[a, b] \subset \mathbf{R}$.

4. Suppose $f : \mathbf{R} \to \mathbf{R}$ is continuous, $[a, b] \subset \mathbf{R}$, and $g : [a, b] \to \mathbf{R}$ is Riemann integrable. Prove that $f \circ g$ is Riemann integrable on $[a, b]$.

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Topology

1. Prove that each compact subset of a metric space is closed and bounded.

2. If X is a nonempty set and $x_0 \in X$, we let $\mathcal{D}(x_0) = \{X\} \cup \{S \subset X \mid x_0 \notin S\}$.
   a. Prove that for each nonempty set X and each $x_0 \in X$ the collection $\mathcal{D}(x_0)$ is a topology on X.
   b. Suppose X and Y are sets, $x_0 \in X$, and $y_0 \in Y$. Prove the following.
   $f : (X, \mathcal{D}(x_0)) \to (Y, \mathcal{D}(y_0))$ is continuous iff either $f(x_0) = y_0$ or f is constant.

3. Given topological spaces X and Y, prove that $X \times Y$ with the product topology is connected iff X and Y are connected.

4. A subset A of a topological space X is called a *retract* of X if there exists a continuous map $r : X \to A$ such that $r(a) = a$ for each $a \in A$. Prove each of the following.
   a. If X is a Hausdorff space and A is a retract of X, then A is closed in X.
   b. The two-point set $\{-1, 1\} \subset \mathbf{R}$ (with the usual topology) is not a retract of $\mathbf{R}$.

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Comprehensive Examination–Option III

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Applied Analysis

1. Determine all values $\alpha \in \mathbf{R}$ such that each solution of $x^2y'' + 5xy' + \alpha y = 0$ satisfies

$$\lim_{x\to\infty} y = 0.$$

2. Solve the linear system $X' = AX$ with

$$A = \begin{pmatrix} -3 & 2 & -1 \\ 1 & -2 & 2 \\ 6 & -6 & 5 \end{pmatrix} \qquad \text{and initial condition } X(0) = X_0 = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}.$$

3. Suppose $A \subset \mathbf{R}$ and $f : A \to \mathbf{R}$. Complete the definition: f is *uniformly continuous* on A if ..., and using your definition prove that if $a > 1/2$, then the function g defined by

$$g(x) = \frac{5x}{2x-1}$$

is uniformly continuous on $[a, \infty)$.

4. State and prove the intermediate value theorem for functions $f : [a, b] \to \mathbf{R}$ where $[a, b] \subset \mathbf{R}$.

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Comprehensive Examination–Option III

2007 Spring

Linear Programming

1. Let P be the following linear programming problem.

$$\begin{array}{ll} \text{minimize} & z = 30x_1 + 12x_2 + 10x_3 \\ \text{subject to} & -x_1 + 3x_2 - x_3 \geq 1 \\ & 6x_1 + 2x_2 + 2x_3 \geq 6 \\ & x_1 - 2x_2 + x_3 \geq 0 \\ & x_j \geq 0, \quad j = 1, 2, 3 \end{array}$$

   a. State the dual of problem P; call it problem Q.
   b. Use the simplex method to solve problem Q.
   c. Use the final tableau from part b. to find an optimal solution of problem P.
   d. Suppose the objective function of problem P is changed to

$$\text{minimize } z = 30x_1 + 10x_2 + 12x_3.$$

   Apply techniques of sensitivity analysis **to problem Q** to find an optimal solution to the modified problem P.
   (No credit for simply changing the numbers and using the simplex method.)

2. In the following $\mathbf{c}$ is $n \times 1$, A is $m \times n$, and $\mathbf{c}^t$ is the transpose of $\mathbf{c}$.
   Use linear programming duality theorems to prove the following.
   If $\mathbf{c}^t\mathbf{x} \geq 0$ for all $\mathbf{x} \in \{\mathbf{x} \mid A\mathbf{x} \geq \mathbf{0} \text{ and } \mathbf{x} \geq \mathbf{0}\}$, then there exists a vector $\mathbf{y}$ such that $A^t\mathbf{y} \leq \mathbf{c}$ and $\mathbf{y} \geq \mathbf{0}$.

Linear Programming—continued

3. In the following let $\mathbf{b}$ be an $m \times 1$ vector, let $\mathbf{c}$ be an $n \times 1$ vector, let A be $m \times n$, and let α be a fixed real number.
   a. Prove that $\{\mathbf{x} \in \mathbf{R}^n \mid \mathbf{c}^t\mathbf{x} \geq \alpha\}$ is a convex subset of $\mathbf{R}^n$.
   b. Prove that the intersection of any collection of convex sets is a convex set.
   c. Suppose the problem

$$\begin{aligned} \text{P} : \quad & \text{minimize} \quad \mathbf{c}^t\mathbf{x} \\ & \text{subject to} \quad A\mathbf{x} \geq \mathbf{b} \\ & \qquad\qquad\quad \mathbf{x} \geq \mathbf{0} \end{aligned}$$

   has an optimal solution. Prove that the set of all optimal solutions of problem P is a convex subset of $\mathbf{R}^n$.

4. Six employees are available to perform six jobs. The time it takes each person to perform each job is given in the following table. Determine the assignment of employees to jobs that minimizes the total time required to perform the six jobs.

Time (hours)

Person	Job 1	Job 2	Job 3	Job 4	Job 5	Job 6
1	3	5	2	6	9	8
2	6	1	8	4	2	5
3	5	3	7	9	4	2
4	2	3	9	5	8	1
5	8	1	6	3	2	7
6	5	6	5	9	7	4

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Comprehensive Examination–Option III

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Numerical Analysis

1. Let $g(x) = 2 + \ln(x)$.
   a. Prove that there exist exactly two positive solutions of the equation $x = g(x)$.
   b. Consider the use of fixed-point iteration on $g(x)$ to approximate α, the greater of the two solutions. Find an interval $[a, b]$ such that for any $x_0 \in [a, b]$ the sequence of iterates $((x_n)_{n=0}^{\infty}$ where $1 \leq n \Rightarrow x_n = g(x_{n-1}))$ converges to α.
   Justify your answer.

2. Consider $f : [0, 1] \to \mathbf{R}$ by $f(x) = x^3$.
   a. Find the polynomial $p(x)$ of degree ≤ 2 which interpolates f on the set $\{0, 1/2, 1\}$.
   b. Prove that $0 \leq x \leq 1 \Rightarrow |f(x) - p(x)| \leq \sqrt{3}/36$.

3. The 3×3 Hilbert matrix H_3 is defined as follows.

$$H_3 = \begin{pmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{pmatrix}$$

   Prove that H_3 is positive definite.

4. Write an efficient algorithm to solve an $n \times n$ linear system $A\mathbf{x} = \mathbf{y}$ where the matrix A has three nonzero diagonals as follows.

$$A = \begin{pmatrix} a_1 & 0 & d_3 & & & & & \\ b_1 & a_2 & 0 & d_4 & & & & \\ & b_2 & a_3 & 0 & d_5 & & & \\ & \cdot & \cdot & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & & b_{n-3} & a_{n-2} & 0 & d_n \\ & & & & & b_{n-2} & a_{n-1} & 0 \\ & & & & & & b_{n-1} & a_n \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-2} \\ x_{n-1} \\ x_n \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \cdot \\ \cdot \\ \cdot \\ y_{n-2} \\ y_{n-1} \\ y_n \end{pmatrix}$$

   Assume that A is nonsingular and that pivoting is unnecessary.
   You should first convert to an upper-triangular system and then apply back-substitution.
   Fully exploit the sparsity pattern.

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Comprehensive Examination–Option III

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Probability

1. A friend is diagnosed as having a very rare type of cancer. The friend is very upset. Since the diagnostic tests are not perfect, due to the possibility of giving false positive and false negative results, it is possible that the diagnosis is incorrect.

   Let C be the event that the friend has cancer and let $+$ be the event that an individual tests positive. Assume $P(C) = 1/1,000,000$ and $P(+|C) = 0.99$ and $P(+|\overline{C}) = 0.1$.

   a. What is the probability that the diagnostic test produces a false negative result?
   b. What is the probability that the diagnostic test produces a false positive result?
   c. Find the probability that the friend has cancer given a positive test result.
   d. If the friend later finds out he does not have cancer, is this surprising?

2. A patient is examined for heart and lung disease using scores X and Y, respectively. A score has value 0 if there is no impairment, 1 if there is mild impairment, and 2 if there is severe impairment. Suppose that the distributions of X and Y in the population are as follows.

x	0	1	2
$P(X = x)$	0.6	0.3	0.1

   and

y	0	1	2
$P(Y = y)$	0.5	0.3	0.2

   Let $T = X + Y$ be the total score for the heart and lung impairment for a patient.

   a. List each of the nine possible heart and lung score combinations. For each outcome compute the value of the total impairment score T and the associated probability, assuming independence between X and Y.
   b. Determine the distribution of T.
   c. Calculate the $E[T]$ and the $Var(T)$.
   d. Suppose the heart and lung impairment are not independent but are positively correlated. Would the positive correlation change $E[T]$? Would it change $Var(T)$? Explain.

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Probability—continued

3. Let the random variable X have the probability density function

$$f(x) = \frac{1}{\beta}\left(1 - \left|\frac{x-\alpha}{\beta}\right|\right) \qquad -\beta + \alpha < x < \beta + \alpha$$

where $-\infty < \alpha < \infty$ and $\beta > 0$.

a. Demonstrate that $f(x)$ is a density function and sketch it.
b. Find the cumulative distribution function $F(x)$.
c. Find the mean of X.
d. Find the variance of X.

4. Suppose $X_1,\ X_2$, and X_3 are independent standard normal random variables with density

$$f(x_i) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x_i^2}{2}\right) \qquad -\infty < x_i < \infty$$

for $i = 1,\ 2,\ 3$.

Let $Y_1 = X_1,\ Y_2 = (X_1 + X_2)/2$, and $Y_3 = (X_1 + X_2 + X_3)/3$.

a. Determine the joint density $F_{Y_1,Y_2,Y_3}(y_1, y_2, y_3)$.
b. Determine the marginal density of $f_{Y_3}(y_3)$.
c. Argue that your answer to part b. is correct using your knowledge of the normal distribution and by computing $E[Y_3]$ and $Var(Y_3)$.