Department of Mathematics and Computer Science

Comprehensive Examination–Option I

2005 Spring

Algebra

1. Let G be a group of order pq, where p and q are distinct primes, and suppose that H and K are normal subgroups of G of orders p and q, respectively. Prove each of the following.
   a. $H \cap K = \{e\}$, where e is the identity element of G.
   b. $G = HK$.
   c. G is cyclic.

2. Let $\mathbf{Z}$ be the ring of integers and suppose $\phi : \mathbf{Z} \to R$ is a homomorphism of $\mathbf{Z}$ onto the ring R. Prove that R is isomorphic as a ring either to $\mathbf{Z}_n$ for some positive integer n, or to $\mathbf{Z}$.

3. Let $F = \mathbf{Z}_2$, the field of integers mod 2.
   a. Prove that $p(x) = x^2 + x + 1$ is the only irreducible polynomial of degree 2 in $F[x]$.
   b. Consider $E = F[x]/(p(x))$ as an extension field of F. Write out explicit addition and multiplication tables for E.
   c. Prove that if $\alpha \in E$ but $\alpha \notin F$, then $x^2 + x + \alpha$ is irreducible in $E[x]$.

4. Let W_1 and W_2 be subspaces of a vector space V. Prove each of the following.
   a. $W_1 + W_2$ is a subspace of V.
   c. If V is finite dimensional, then $\dim W_1 + \dim W_2 = \dim(W_1 \cap W_2) + \dim(W_1 + W_2)$.

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2005 Spring

Complex Analysis

1. Find the image under the transformation $w = \dfrac{z-1}{z+1}$ of
   a. $\{z \in \mathbf{C} \mid |z+2| = 1\}$;
   b. the imaginary axis.

2. Find the maximum value of $|f(z)| = |z(z-i)(z-2i)|$ on the closed unit disk $D = \{z \in \mathbf{C} \mid |z| \le 1\}$, and find a point at which this maximum is attained.

3. Evaluate
$$\int_0^\infty \frac{x^2}{(x^2+1)^3}\,dx$$
using the method of residues.

4. Let $f(z) = \dfrac{z^4(z^2-1)}{(z^2+1)(z^2+9)^3}$. Evaluate
$$\int_C \frac{f'(z)\,dz}{f(z)}$$
where C is the circle $\{z \in \mathbf{C} \mid |z| = 2\}$ traversed counterclockwise.

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Real Analysis

1. a. Prove that each convergent sequence in a metric space is bounded.
   b. Prove that if $(a_n)_{n=0}^{\infty} \to a$ and $(b_n)_{n=0}^{\infty} \to b$ in $\mathbf{R}$, then $(a_n b_n)_{n=0}^{\infty} \to ab$ in $\mathbf{R}$.

2. Let $f : \mathbf{R} \to \mathbf{R}$ be differentiable on $\mathbf{R}$. Prove each of the following.
   a. $f(-x) = f(x)\ \forall x \in \mathbf{R} \ \Rightarrow\ f'(-x) = -f'(x)\ \forall x \in \mathbf{R}$.
   b. $f(-x) = -f(x)\ \forall x \in \mathbf{R} \ \Rightarrow\ f'(-x) = f'(x)\ \forall x \in \mathbf{R}$.

3. Determine which of the following functions are uniformly continuous on the given set, and justify your answers.

   a) $y = \dfrac{x-1}{x+1}$ on $[0, \infty)$ b) $y = \dfrac{e^x}{x}$ on $[1, 100]$ c) $y = \dfrac{e^x}{x}$ on $(1, 2)$

4. Let $f(x) = \lim_{n\to\infty} f_n(x)\ \forall x \in \mathbf{R}$ with $f_n : \mathbf{R} \to \mathbf{R}$ differentiable on $\mathbf{R}$ and $|f_n'(x)| \le \frac{1}{2}\ \forall x \in \mathbf{R}$ and $n = 1, 2, 3, \ldots$. Prove that $|f(x) - f(y)| \le |x - y|\ \forall x, y \in \mathbf{R}$.

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Topology

1. Let $A \subset X$ have the relative topology $\mathcal{T}_A$ induced by the topology $\mathcal{T}$ on X.
   Prove or disprove: $B \subset A$ is compact in A iff $B \subset A$ is compact in X.

2. Let X be a compact space, Y be a Hausdorff space, and $f : X \to Y$ be a continuous surjection. Prove that $f^{-1}(V)$ is open in X if and only if V is open in Y.

3. Let A be a closed subset of a normal space $(X, \mathcal{T})$, and assume $A \subset U$ for some open subset $U \subset X$. Prove that there exists an open set V with $A \subset V \subset \overline{V} \subset U$.

4. Let $\pi : X \times Y \to X$ be the projection given by $\pi(x, y) = x \ \forall (x, y) \in X \times Y$ and let Y be compact. Prove: $\pi(A)$ is closed in X for each closed $A \subset X \times Y$.
   ($X \times Y$ has the usual product topology.)

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Comprehensive Examination–Option III

2005 Spring

Applied Analysis

1. a. Prove that each convergent sequence in a metric space is bounded.
   b. Prove that if $(a_n)_{n=0}^{\infty} \to a$ and $(b_n)_{n=0}^{\infty} \to b$ in $\mathbf{R}$, then $(a_n b_n)_{n=0}^{\infty} \to ab$ in $\mathbf{R}$.

2. Let $f : \mathbf{R} \to \mathbf{R}$ be differentiable on $\mathbf{R}$. Prove each of the following.
   a. $f(-x) = f(x)\ \forall x \in \mathbf{R} \;\Rightarrow\; f'(-x) = -f'(x)\ \forall x \in \mathbf{R}$.
   b. $f(-x) = -f(x)\ \forall x \in \mathbf{R} \;\Rightarrow\; f'(-x) = f'(x)\ \forall x \in \mathbf{R}$.

3. Newton's law of cooling states that the rate at which a body cools is proportional to the difference between the temperature of the body and that of the medium in which it is situated. Suppose a body of temperature 80°F is placed at time $t = 0$ into a medium whose temperature is maintained at 50°F, and suppose that at the end of five minutes the body has cooled to a temperature of 70°F. Notice that Newton's law of cooling gives a differential equation of the form $y' = k(y - T)$.
   a. What is the temperature of the body at the end of twelve minutes?
   b. When will the temperature of the body be 60°F?

4. $(1 - x^2)y'' - 2xy' + r(r+1)y = 0$ is called Legendre's equation of order r.
   a. Find all singular points of Legendre's equation (of order r) and classify them as regular or irregular.
   b. Find the first four nonzero terms in the Maclaurin series expansion of the solution to Legendre's equation (of order r) with initial conditions $y(0) = 1$ and $y'(0) = 0$.

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Comprehensive Examination–Option III

2005 Spring

Linear Programming

1. Let P and Q be the following linear programming problems.

$$\text{P} \quad \begin{cases} \text{maximize } \mathbf{c}^t\mathbf{x} \\ \qquad A\mathbf{x} \leq \mathbf{b} \end{cases} \qquad \text{Q} \quad \begin{cases} \text{minimize } \mathbf{c}^t\mathbf{x} \\ \qquad A\mathbf{x} \geq \mathbf{b} \end{cases}$$

where A is an $m \times n$ matrix, $\mathbf{b}$ is a constant vector with m components, and $\mathbf{c}$ is a constant vector with n components. $\mathbf{c}^t$ is the transpose of the column vector $\mathbf{c}$.

a. Write the duals to problems P and Q.

b. Suppose both problems P and Q have feasible solutions.
Prove: If problem P has a finite optimal solution, then so does problem Q.

c. Suppose both problems P and Q have finite optimal solutions, and suppose that $\mathbf{x}$ is feasible for problem P and $\hat{\mathbf{x}}$ is feasible for problem Q.
Prove that $\mathbf{c}^t\mathbf{x} \leq \mathbf{c}^t\hat{\mathbf{x}}$.

Linear Programming—continued

2. Consider the following linear programming problem.

$$\begin{aligned} \text{minimize } z = -101x_1 + 87x_2 + 23x_3 & \\ \text{subject to} \quad 6x_1 - 13x_2 - 3x_3 &\leq 11 \\ 6x_1 + 11x_2 + 2x_3 &\leq 45 \\ x_1 + 5x_2 + x_3 &\leq 12 \\ x_1,\ x_2,\ x_3 &\geq 0 \end{aligned}$$

Using techniques of sensitivity analysis answer the following questions. Note that each of the following questions is independent of its predecessors in the sense that after each question that involves changing a number, the number should be re-set to its original value before answering the next question.

a. By how much can the right hand side of the second constraint increase and decrease without changing the optimal basis?

b. Would the current basis remain optimal if a new variable x_7 were added to the problem with objective function coefficient $c_7 = 46$ and coefficients of x_7 in constraints 1, 2, and 3 being 12, -14, and 15, respectively?

c. If the objective function is replaced with $z = -101x_1 + (87+\theta)x_2 + 23x_3$, find an optimal solution vector of this revised problem for each value of θ in the interval $[0, 0.5]$.

In solving parts a., b., and c. use the fact that the final tableau for the original problem is the following (where row 4 is the objective function row, and -372 is the optimal value of the objective function).

x_1	x_2	x_3	x_4	x_5	x_6	
1	0	0	1	-2	7	5
0	1	0	-4	9	-30	1
0	0	1	19	-43	144	2
0	0	0	12	4	5	372

Linear Programming—continued

3. Five employees are available to perform five jobs. The time it takes each person to perform each job is given by the following table. Determine the assignment of employees to jobs that minimizes the total time required to perform the five jobs. (Note: In the table an asterisk (∗) indicates that the person cannot do that particular job.)

	Time (hours)				
Person	Job 1	Job 2	Job 3	Job 4	Job 5
1	22	18	30	18	21
2	18	∗	27	22	20
3	26	20	28	28	28
4	16	22	∗	14	20
5	21	∗	25	28	29

4. a. Solve the following linear programming problem using the dual simplex method.

$$\begin{aligned}
\text{minimize} \quad & 3x_1 + x_2 + 5x_3 \\
\text{subject to} \quad & -x_1 - x_2 + x_3 \le -5 \\
& 2x_1 + 2x_2 - 4x_3 \le -8 \\
& -2x_1 - x_2 - x_3 \le -10 \\
& x_1,\ x_2,\ x_3 \ge 0
\end{aligned}$$

   b. From the tableau of part a. determine the optimal solution of the dual of the given problem.

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Comprehensive Examination–Option III

2005 Spring

Numerical Analysis

1. a. Prove that the equation $x^2 - 1 = \sin x$ has exactly two real solutions.
   b. Consider using fixed point iteration on $g(x) = \sqrt{1 + \sin x}$ to approximate the positive solution, α. Show that for any initial approximation $x_0 \in [0, \pi/2]$ the iteration sequence will converge to α.

2. Consider the difference formula

$$f'(x_0) = \frac{-f(x_0 + 2h) + 4f(x_0 + h) - 3f(x_0)}{2h} + \frac{1}{3}f'''(x_0)h^2 + O(h^3).$$

Observe that

$$f'(x_0) = \frac{-f(x_0 + 2h) + 4f(x_0 + h) - 3f(x_0)}{2h}$$

is a $O(h^2)$ approximation of $f'(x_0)$. Use the formula to construct a $O(h^3)$ approximation of $f'(x_0)$ that involves $f(x_0)$, $f(x_0 + h)$, $f(x_0 + 2h)$, and $f(x_0 + 4h)$.

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Numerical Analysis—continued

3. Let L be a lower-triangular matrix of the form

$$\begin{pmatrix} a_1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ b_1 & a_2 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ c_1 & b_2 & a_3 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & c_2 & b_3 & a_4 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_{n-3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & b_{n-3} & a_{n-2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & c_{n-3} & b_{n-2} & a_{n-1} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & c_{n-2} & b_{n-1} & a_n \end{pmatrix}$$

with $a_i \neq 0, \; i = 1, 2, \ldots, n$.

a. Construct an efficient algorithm to solve $(LL^t)\mathbf{x} = \mathbf{y}$ where $\mathbf{y} = (y_1, y_2, \ldots, y_n)^t$ is a given n-vector and $\mathbf{x} = (x_1, x_2, \ldots, x_n)^t$ is the solution vector.

b. Determine the precise number of floating point operations in your algorithm. Include all additions, subtractions, multiplications, and divisions.

4. Suppose that $\| \; \|$ is a vector norm on $\mathbf{R}^n$ and we define a matrix norm as follows. For each $n \times n$ real matrix A we let

$$\|A\| = \max\{\|A\mathbf{u}\| \mid \|\mathbf{u}\| = 1\}.$$

Also for each invertible $n \times n$ real matrix A we define the condition number (with respect to $\| \; \|$) by $\text{cond}_{\| \; \|}(A) = \|A\|\|A^{-1}\|$.

a. Prove that if A and B are $n \times n$ real matrices, then $\|AB\| \leq \|A\|\|B\|$.

b. Prove that if A and B are invertible $n \times n$ real matrices, then

$$\text{cond}_{\| \; \|}(AB) \leq \text{cond}_{\| \; \|}(A)\text{cond}_{\| \; \|}(B).$$

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Comprehensive Examination–Option III

2005 Spring

Probability

1. An urn contains five red balls, six blue balls, and eight green balls. If a set of three balls is randomly selected without replacement, what is the probability that the balls will be
   a. of the same color?
   b. of different colors?

   Repeat parts a. and b. above under the assumption that whenever a ball is selected its color is noted, and it is then replaced into the urn before the next selection. This is known as sampling with replacement.

2. Suppose X and Y are random variables that assume the values x and y, where $x = 1$ or $x = 2$, and $y = 1$ or $y = 2$ or $y = 3$ or $y = 4$, with probabilities given by the following table.

	y			
x	1	2	3	4
1	1/4	1/8	1/16	1/16
2	1/16	1/16	1/4	1/8

   a. Determine the marginal p.m.f. of X, $f_X(x)$, and the marginal p.m.f. of Y, $f_Y(y)$. Sketch these p.m.f.s and describe their shapes.
   b. Calculate $E[X]$, $E[Y]$, and $E[Y \mid X = x]$, for $x = 1$ and for $x = 2$.
   c. Calculate $E[E[Y \mid X]]$. Compare $E[Y]$ and $E[E[Y \mid X]]$.
   d. Calculate $Var(X)$ and $Var(Y)$.

Probability—continued

3. Suppose a test for diagnosing heart disease has a 0.90 probability of positively identifying the disease D when it is present. Suppose the test wrongly positively identifies the disease with probability 0.02 when the disease is not present. From statistical data it is known that in a certain population 5 of 1000 persons have the disease. An individual is randomly chosen from this population and is given the test.
   a. Calculate the probability that the test is positive, $P(+)$.
   b. Calculate the probability that the individual actually suffers from the disease D if the test turns out to be positive, $P(D|+)$.
   c. Calculate the probability that the individual actually does not suffer from the disease D^c if the test turns out to be positive, $P(D^c|+)$.
   d. Is the result for $P(D^c|+)$ surprising? Explain.

4. Let X, Y, and Z be independent random variables having the identical density function

$$f(x) = e^{-x} \qquad 0 < x < \infty.$$

   a. Derive the joint density functions of $U = X + Y$, $V = X + Z$, and $W = Y + Z$.
   b. Are U, V, and W independent?