Department of Mathematics and Computer Science

Comprehensive Examination–Option I

2006 Autumn

Algebra

1. Let G be a finite group of order n with multiplicative identity e. Prove the following.
   a. If G is abelian and n is odd, then the product of all the elements of G is e.
   b. If n is even, then there is an element $a \in G$ such that $a \neq e$ and $a^2 = e$.

2. Let $\varphi : G \to G'$ be a group homomorphism with kernel K, and suppose N' is a normal subgroup of G'. Prove that $N = \varphi^{-1}(N') = \{x \in G \mid \varphi(x) \in N'\}$ is a normal subgroup of G containing K.

3. a. Prove that $I = \mathbf{Z} \times \{0\} = \{(a, 0) \mid a \in \mathbf{Z}\}$ is an ideal of $R = \mathbf{Z} \times \mathbf{Z}$.
   b. Determine whether I is a prime ideal or a maximal ideal of R.
   Justify your answers.

4. Let U and W be subspaces of a vector space V over a field F. Prove the following.
   a. If $U \cap W = \{\mathbf{0}\}$, then for all nonzero vectors $\mathbf{u} \in U$ and $\mathbf{w} \in W$ the set $\{\mathbf{u}, \mathbf{w}\}$ is linearly independent.
   b. If V is finite dimensional and $\dim_F(V) < \dim_F(U) + \dim_F(W)$, then $U \cap W \neq \{\mathbf{0}\}$.

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Complex Analysis

1. Evaluate
$$\int_C \frac{e^z\,dz}{(z^2-4)(z^2+9)}$$
where C is the circle of radius 3 centered at $z_0 = 1+i$ traversed once clockwise.

2. Let $f(z) = z^2 + (1+i)z$ and $D = \{z \in \mathbf{C} \mid |z| \le 1\}$. Find the maximum value of $|f(z)|$ on D and an explicit point $z_0 \in D$ with $|f(z_0)|$ maximal.

3. Find the Laurent series expansion of
$$f(z) = \frac{6z^2}{z^2 - z - 2}$$
in the annulus $\{z \in \mathbf{C} \mid 1 < |z| < 2\}$.

4. Use the residue theorem to prove that
$$\int_0^\infty \frac{x^2\,dx}{x^4+1} = \frac{\pi}{2\sqrt{2}}.$$

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Real Analysis

1. a. Complete the following definition.
   A sequence (x_n) of real numbers *converges to a limit* L if ... .
   b. Use the definition to prove:
   If $\forall n\ 0 \leq x_n$ and $(x_n) \to a$, then $(\sqrt{x_n}) \to \sqrt{a}$.

2. Let $f : \mathbf{R} \to \mathbf{R}$ be defined by

$$f(x) = \sum_{n=1}^{\infty} \frac{3^n x^2 \sin(n^3 x)}{(n+1)^n} \text{ for each } x \in \mathbf{R}.$$

   Prove that f is continuous on $\mathbf{R}$.

3. Suppose $0 < M < 1$, $f : \mathbf{R} \to \mathbf{R}$, $\forall x \in [0, 1]\ |f'(x)| \leq M$, and $f(1/2) = 1/2$.
   Prove that f maps $[0, 1]$ into $[0, 1]$.

4. Prove: If $[a, b] \subset \mathbf{R}$ and $f : [a, b] \to \mathbf{R}$ is continuous, then the Riemann integral

$$\int_a^b f(x)\, dx$$

   exists.

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Topology

1. Suppose (X, d) is a metric space.
   a. Prove that if $x \in X$ and U is an open set in X containing x, then there exists an open set V in X such that $x \in V \subset \overline{V} \subset U$.
   b. Use a. to prove that if $x \in X$ and F is a nonempty closed set in X not containing x, then there exist disjoint open sets U_1 and U_2 in X such that $x \in U_1$ and $F \subset U_2$.

2. Let X be a topological space and $A \subset X$. The *boundary* ∂A of A is defined to be $\overline{A} \cap \overline{(X - A)}$ where $\overline{A}$ denotes the closure of A. Prove that $\overline{A} = A \cup \partial A$.

3. Let A and B be nonempty connected subsets of a topological space X. Prove that $\overline{A} \cup \overline{B}$ is connected if and only if $\overline{A} \cap \overline{B} \neq \emptyset$.

4. Let $f : X \to Y$ be a continuous map from a compact space X into a Hausdorff space Y. Prove the following.
   a. f is a closed map.
   b. If f is a bijection, then f is a homeomorphism.

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Applied Analysis

1. Solve the linear system

$$\begin{aligned} x' &= x \phantom{{}-y} + z \\ y' &= 2x - y + 5z \\ z' &= x - y + 3z \end{aligned}$$

with initial conditions $x(0) = 1,\ y(0) = -1,\ z(0) = 1.$

2. Find the first five nonzero terms in the Maclaurin series expansion for the solution of

$$y'' + (\sin x^2)y' - x^3 y = 0$$

with initial conditions $y(0) = 1$ and $y'(0) = 2$.

3. a. Complete the following definition.
   A sequence (x_n) of real numbers *converges to a limit* L if . . . .
   b. Use the definition to prove:
   If $\forall n\ 0 \le x_n$ and $(x_n) \to a$, then $(\sqrt{x_n}) \to \sqrt{a}$.

4. Let $f : \mathbf{R} \to \mathbf{R}$ be defined by

$$f(x) = \sum_{n=1}^{\infty} \frac{3^n x^2 \sin(n^3 x)}{(n+1)^n} \quad \text{for each } x \in \mathbf{R}.$$

Prove that f is continuous on $\mathbf{R}$.

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Comprehensive Examination–Option III

2006 Autumn

Linear Programming

1. Consider the following problem P.

$$\begin{aligned} \text{P}: \quad & \text{minimize } z = 2x_1 + 4x_2 \\ & \text{subject to} \quad 3x_1 + 2x_2 = 14 \\ & \qquad 2x_1 - 4x_2 \geq \ 2 \\ & \qquad 4x_1 + 8x_2 \leq 29 \\ & \qquad\qquad x_j \geq \ 0, \ j = 1, 2 \end{aligned}$$

   a. Solve problem P using the two phase simplex method; *i.e.*, using artificial variables.
   b. State the dual of problem P, and use the final tableau from part a. to find the optimal values of the dual variables.

2. Let P and $\overline{\text{P}}$ be the following linear programming problems.

$$\text{P} \quad \begin{cases} \text{minimize } \mathbf{c}^t\mathbf{x} \\ A\mathbf{x} \geq \mathbf{b} \\ \mathbf{x} \geq \mathbf{0} \end{cases} \qquad\qquad \overline{\text{P}} \quad \begin{cases} \text{minimize } \mathbf{c}^t\mathbf{x} \\ A\mathbf{x} \geq \mathbf{b} + \mathbf{d} \\ \mathbf{x} \geq \mathbf{0} \end{cases}$$

   where A is an $m \times n$ matrix and $\mathbf{c}^t$ is the transpose of the column vector $\mathbf{c}$.
   ($\mathbf{b}$ and $\mathbf{d}$ are constant vectors with m components.)
   Prove that if $\mathbf{x}^*$ is the optimal solution of P, $\mathbf{y}^*$ is the optimal solution of the dual of P, and $\overline{\mathbf{x}}^*$ is the optimal solution of $\overline{\text{P}}$, then

$$\mathbf{c}^t\overline{\mathbf{x}}^* \geq \mathbf{c}^t\mathbf{x}^* + \mathbf{d}^t\mathbf{y}^*.$$

   (Hint: Write out the duals of problems P and $\overline{\text{P}}$.)

Linear Programming—continued

3. State and prove the Complementary Slackness Theorem for the pair of linear programming problems of the form

P. minimize $\mathbf{c}^t\mathbf{x}$ subject to $A\mathbf{x} \geq \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$

Q. maximize $\mathbf{b}^t\mathbf{y}$ subject to $A^t\mathbf{y} \leq \mathbf{c}$, $\mathbf{y} \geq \mathbf{0}$

where A is an $m \times n$ matrix.

4. Consider the following linear programming problem.

$$
\begin{aligned}
\text{minimize } & z = -3x_1 - 13x_2 - 13x_3 \\
\text{subject to } & x_1 + x_2 \leq 7 \\
& x_1 + 3x_2 + 2x_3 \leq 15 \\
& 2x_2 + 3x_3 \leq 9 \\
& x_1,\ x_2,\ x_3 \geq 0
\end{aligned}
$$

a. Use the simplex method to find optimal values of the variables.

Use techniques of sensitivity analysis to answer parts b.–d.
Note that each of these questions is independent of its predecessors in the sense that after each question that involves changing a number, that number should be re-set to its original value before answering the next question.

b. By how much can the number on the right side of the first constraint increase and decrease without changing the current optimal basis?

c. What is the optimal solution of the linear programming problem obtained by decreasing the objective coefficient of x_2 from -13 to -28?

d. Determine the optimal solution of the linear programming problem obtained by adding the constraint $x_1 + 3x_2 \leq 7$.

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Numerical Analysis

1. Consider the equation $e^x = 3x + 2$.
   a. Show that there exist precisely two real solutions of the equation $e^x = 3x + 2$, one negative and one positive.
   b. Suppose z is the positive solution; find an approximation x of z such that $|z - x| < 10^{-6}$.
   c. Show that your approximation x of z is in fact within 10^{-6} of z.

2. Consider the following formula for approximating the derivative $f'(a)$.

$$f'(a) \approx A(h) = \frac{f(a-h) - 8f(a-h/2) + 8f(a+h/2) - f(a+h)}{6h}$$

   a. Show that if the fifth derivative $f^{(5)}$ of f is continuous near a, then for small h one has
$$f'(a) - A(h) \approx \frac{f^{(5)}(a)}{480} h^4,$$
   so that the truncation error is $O(h^4)$.
   b. Consider $f(x) = e^x$ and let $h = 0.1$.
   Compute accurately the quantity $f'(1) - A(0.1)$, and compare with the approximation in part a. above.

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Numerical analysis—continued

3. Let

$$A = \begin{pmatrix} 6 & -1 & 0 & 2 \\ -1 & 6 & -1 & 0 \\ 0 & -1 & 6 & -1 \\ 2 & 0 & -1 & 6 \end{pmatrix}.$$

Suppose we attempt to solve the equation $A\mathbf{x} = \mathbf{b}$ by the iterative method

$$\mathbf{x}^{(k)} = M^{-1}N\mathbf{x}^{(k-1)} + M^{-1}\mathbf{b}$$

where M and N come from the following splitting of A.

$$A = M - N = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix} - \begin{pmatrix} -1 & 1 & 0 & -2 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ -2 & 0 & 1 & -1 \end{pmatrix}$$

Explain why, for each $\mathbf{b}$ and for each initial approximation $\mathbf{x}^{(0)}$, the sequence $(\mathbf{x}^{(k)})$ converges to the solution of $A\mathbf{x} = \mathbf{b}$.

4. Let I_n be the $n \times n$ identity matrix.
   a. Prove that $1 \leq \|I_n\|$ for each matrix norm.
   b. Let A be a nonsingular $n \times n$ matrix. Use the result of part a. to prove that $1 \leq \text{Cond}(A)$ with respect to any matrix norm.

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Probability

1. Assume that X is a continuous random variable with probability density function

$$f(x) = \begin{cases} \exp[-(x+2)] & \text{if } -2 < x \\ 0 & \text{if } \quad x \leq -2. \end{cases}$$

   a. Sketch a graph of $f(x)$.
   b. Find the moment generating function of X.
   c. Use the moment generating function from part a. to determine the expected value of X, $E[X]$.
   d. Use the moment generating function from part a. to determine the variance of X, $Var(X)$.

2. Let Y and U be two independent random variables, with $Y \sim N(0,1)$ and

$$P(U = -1) = P(U = 1) = 1/2.$$

   Let $Z = UY$.
   a. Show that $Z \sim N(0,1)$.
   b. Show that Y and Z are uncorrelated.
   c. Show that Y and Z are not independent.

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Probability—continued

3. Suppose the number of automobile accidents a driver will be involved in during a one-year period is a random variable Y having a Poisson distribution with parameter λ, where λ depends on the driver. If we choose a driver at random from some population, we can let λ vary and define a continuous random variable Λ having the probability density f_Λ. Then the conditional density of Y given $\Lambda = \lambda$ is the Poisson density with parameter λ

$$f_{Y|\Lambda}(y|\lambda) = \begin{cases} \dfrac{\lambda^y e^{-\lambda}}{y!} & \text{if } y = 0, 1, 2, \dots \\ 0 & \text{elsewhere.} \end{cases}$$

Suppose we assume that the density f_Λ is a gamma density $\Gamma(\alpha, \beta)$, where

$$f_\Lambda(\lambda) = \begin{cases} \dfrac{\beta^\alpha \lambda^{\alpha-1} e^{-\lambda\beta}}{\Gamma(\alpha)} & \text{if } \lambda > 0 \\ 0 & \text{elsewhere.} \end{cases}$$

a. Determine the joint density $f(\lambda, y)$.
b. Determine the marginal density $f(y)$.
c. Determine the conditional density $f_{\Lambda|y}(\lambda|y)$. Give the name and the parameters of the distribution of $\Lambda|Y = y$.

4. Suppose a point is picked at random in the unit square. If it is known that the point is in the rectangular region bounded by the lines $y = 0,\ y = 1,\ x = 0$, and $x = 1/2$, what is the probability that the point is in

a. the triangular region bounded by the lines $y = 0,\ x = 1/2$, and $x + y = 1$?
b. the triangular region bounded by the lines $y = 0,\ x = 1/2$, and $x = y$?
c. Now, determine two regions in the unit square that are mutually exclusive. Sketch a graph to show the regions. Show that the regions are mutually exclusive using probability.
d. Determine two regions in the unit square that are independent. Sketch a graph to show the regions. Show the regions are independent using probability.