Department of Mathematics and Computer Science

Comprehensive Examination–Option I

2005 Autumn

Algebra

1. Suppose that H is a subgroup of the group G, and $N = \bigcap\{xHx^{-1} \mid x \in G\}$.
   Prove that N is a subgroup of G, and that N is a normal subgroup of G.

2. Suppose G is a group. For each g in G define $I_g : G \to G$ by $I_g(x) = gxg^{-1}\ \forall x \in G$.
   Prove:
   a. I_g is an automorphism of G; *i.e.*, $I_g \in \mathrm{Aut}(G)$.
   b. If $\phi : G \to \mathrm{Aut}(G)$ is defined by $\phi(g) = I_g\ \forall g \in G$, then ϕ is a homomorphism with kernel $Z(G)$, the center of G.

3. Suppose that R is a commutative ring, $a \in R$, $b \in R$, and $I = \{x \in R \mid ax \in bR\}$.
   Prove that I is an ideal in R.

4. Consider the linear transformation $T : \mathbf{R}^4 \to \mathbf{R}^4$ defined by

$$T(a, b, c, d) = (a + b + c + d, c + d, b + d, b + c + 2d).$$

   Find
   a. a basis for the kernel of T;
   b. a basis for the image of T;
   c. the matrix representation of T, if the (ordered) basis

$$\{(1, 1, 1, 1), (0, 1, 1, 1), (0, 0, 1, 1), (0, 0, 0, 1)\}$$

   is used for both the domain and the codomain of T.

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Complex Analysis

1. Find the linear fractional transformation

$$T(z) = \frac{az+b}{cz+d}$$

which maps $-1,\ 1,\ \infty$ to $1,\ \infty,\ -1$, respectively.

2. Prove that $u : \mathbf{R}^2 \to \mathbf{R}$ by $u(x, y) = \cosh x \cos y$ is harmonic in $\mathbf{R}^2$, and find <u>all</u> harmonic conjugates of u.

3. Use the method of residues to show that

$$\int_0^{2\pi} \frac{d\theta}{5 + 4\sin\theta} = \frac{2\pi}{3}.$$

4. Find the Laurent series expansion of

$$f(z) = z^3 \cos\left(\frac{1}{z^2}\right)$$

about the point $z_0 = 0$, determine the largest open set in which the series converges, and evaluate

$$\int_C f(z)\, dz,$$

where C is the circle of radius $1/4$ centered at 0 and traversed once counterclockwise.

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Real Analysis

1. a. Suppose (X, d) and (Y, ρ) are metric spaces, $A \subset X$, (x_k) is a Cauchy sequence in A, and $f : A \to Y$ is uniformly continuous. Prove that $(f(x_k))$ is a Cauchy sequence.

   b. Give an example of a metric space (X, d), a Cauchy sequence (x_k) in X, and a continuous function $f : X \to X$ such that $(f(x_k))$ is not a Cauchy sequence.

2. Suppose $a < b$, $[a, b] \subset \mathbf{R}$, and $f, g : [a, b] \to \mathbf{R}$ are continuous on $[a, b]$ and differentiable in (a, b). Prove that there exists $c \in (a, b)$ such that

$$[f(a) - f(b)]g'(c) = [g(a) - g(b)]f'(c).$$

3. Suppose $a < b$, $[a, b] \subset \mathbf{R}$, and $f : [a, b] \to \mathbf{R}$ is continuous on $[a, b]$. Prove that if $f \geq 0$ on $[a, b]$ and

$$\int_a^b f(x)\,dx = 0,$$

then $f(x) = 0\ \forall x \in [a, b]$.

4. Prove that the series

$$\sum_{n=1}^{\infty} \frac{nx^2}{x^3 + n^3}$$

converges uniformly on each interval $[0, a]$ with $a > 0$.

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Topology

1. Let X be a Hausdorff space.
   a. Prove that each one-point subset of X is closed.
   b. Prove that if x_0 is a limit point of a subset A of X, then each open set containing x_0 contains infinitely many points of A.

2. Recall that a subset A of a topological space X is dense in X if $A \cap U \neq \emptyset$ for each nonempty open set $U \subset X$.
   a. Prove: If A and B are dense in X and A is open in X, then $A \cap B$ is dense in X.
   b. Give an example of a space X and dense subsets A and B of X such that $A \cap B$ is not dense in X.

3. Prove that each compact metric space is a second countable space; *i.e.*, has a countable basis.

4. Let X be a Hausdorff space, $f : X \to X$ be continuous, and $F = \{x \in X \mid f(x) = x\}$. Prove that F is closed in X.

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Comprehensive Examination–Option III

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Applied Analysis

1. Solve $x^2y'' - 2xy' + 2y = 6x^3$ with initial conditions $y(1) = 0$ and $y'(1) = 1$.

2. Consider the differential equation

$$(1 - x^2)y'' - xy' + \alpha^2 y = 0$$

where α is a constant.
   a. Find the associated recurrence relations, and find two linearly independent power series solutions in powers of x for $|x| < 1$.
   b. Show that if α is a nonnegative integer n, then there is a polynomial solution of degree n.

3. Suppose $a < b$, $[a, b] \subset \mathbf{R}$, and $f : [a, b] \to \mathbf{R}$ is continuous on $[a, b]$. Prove that if $f \geq 0$ on $[a, b]$ and

$$\int_a^b f(x)\, dx = 0,$$

then $f(x) = 0 \ \forall x \in [a, b]$.

4. Prove that the series

$$\sum_{n=1}^{\infty} \frac{nx^2}{x^3 + n^3}$$

converges uniformly on each interval $[0, a]$ with $a > 0$.

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Comprehensive Examination–Option III

2005 Autumn

Linear Programming

1. Let A be an $m \times n$ matrix and $\mathbf{b}$ an $m \times 1$ vector. Use linear programming duality theorems to prove Gale's transposition theorem; that is, prove that there exists a vector $\mathbf{x}$ such that $A\mathbf{x} \leq \mathbf{b}$ if and only if for each vector $\mathbf{y}$ satisfying $A^t\mathbf{y} = \mathbf{0}$ and $\mathbf{y} \geq \mathbf{0}$, we have $\mathbf{b}^t\mathbf{y} \geq \mathbf{0}$.

2. Let P_1 be the following linear programming problem.

$$\begin{aligned} \text{minimize } z = -4x_1 + 6x_3 + 10x_4 & \\ \text{subject to} \quad x_1 - 2x_2 + x_3 + 2x_4 &\leq 20 \\ 6x_1 - 4x_2 + 3x_3 + 2x_4 &\leq 200 \\ 3x_1 + 2x_2 + 9x_3 + 12x_4 &\leq 100 \\ x_j \geq 0, \; j = 1,2,3,4 & \end{aligned}$$

a. Using the simplex method find an optimal solution to problem P_1.

b. Let P_2 be the same as P_1 except that z is changed to $z = -4x_1 + c_2x_2 + 6x_3 + 10x_4$. For what range of values of c_2 is the optimal solution to P_1 also optimal for P_2?

c. Let problem P_3 be the same as P_1 except that the right hand sides of the inequalities are changed from 20, 200, 100 to $b_1,\ b_2,\ b_3$, respectively. If the optimal solution to problem P_3 is $x_1 = 36,\ x_2 = 6,\ x_6 = 48,\ x_3 = x_4 = x_5 = x_7 = 0$, find $b_1,\ b_2$, and b_3.

Linear Programming—continued

3. A manufacturing company has plants in cities A, B, C, D, and E. The company distributes its product to dealers in cities U, V, W, X, and Y. On a particular day the company has 10 units of its product in A, 15 in B, 30 in C, 5 in D, and 10 in E. It plans to ship 15 units to U, 10 to V, 10 to W, 5 to X, and 30 to Y, following orders received from dealers. The unit shipping costs are given in the following table.

From↓ To→	U	V	W	X	Y
A	3	6	8	11	5
B	1	9	3	2	7
C	4	2	8	25	15
D	9	1	4	9	8
E	2	4	2	11	1

Determine a shipping program that fulfills the required orders and minimizes the total shipping cost.

4. Let LP be the following linear programming problem.

$$\begin{aligned} \text{minimize } & \mathbf{c}^t\mathbf{x} \\ \text{subject to } & A\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

where A is $m \times n$, $\mathbf{c}$ is $n \times 1$, and $\mathbf{b}$ is $m \times 1$.

a. State the dual of problem LP.

b. Prove that if $\mathbf{x}$ is feasible for problem LP, and if there exist λ and μ such that

$$\begin{aligned} A^t\lambda + \mu &= \mathbf{c} \\ \mu^t\mathbf{x} &= \mathbf{0} \\ \mu &\geq \mathbf{0}, \end{aligned}$$

then $\mathbf{x}$ is optimal for problem LP and λ is optimal for the dual of LP.

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Numerical Analysis

1. Suppose $a < b$ and $f : [a, b] \to \mathbf{R}$ is a continuous function with a unique zero, z, in the interval $[a, b]$. Assume $0 < \varepsilon$ and consider using the method of bisection to approximate this z with x such that $|z - x| < \varepsilon$. Determine a formula which expresses in terms of a, b, and ε the minimum number of iterations of the bisection algorithm which will assure that $|z - x| < \varepsilon$.

2. Consider $f : [1, 2] \to \mathbf{R}$ by $f(x) = \sqrt[4]{x}$.
   a. Find the polynomial $p(x)$ of degree ≤ 2 which interpolates f on the set $\{1, 3/2, 2\}$.
   b. Prove that $1 \leq x \leq 2 \;\Rightarrow\; |f(x) - p(x)| < 7\sqrt{3}/4608$.

3. Let A be an $n \times n$ pentadiagonal symmetric positive definite matrix of the form

$$A = \begin{pmatrix} a_1 & 0 & b_1 & & & & \\ 0 & a_2 & 0 & b_2 & & & \\ b_1 & 0 & a_3 & 0 & b_3 & & \\ & \cdot & \cdot & \cdot & \cdot & \cdot & \\ & & b_{n-4} & 0 & a_{n-2} & 0 & b_{n-2} \\ & & & b_{n-3} & 0 & a_{n-1} & 0 \\ & & & & b_{n-2} & 0 & a_n \end{pmatrix}.$$

   a. Construct an efficient algorithm which generates a lower triangular matrix L such that $A = LL^t$.
   That is, construct the vectors d, e, and f such that

$$A = LL^t \quad \text{where} \quad L = \begin{pmatrix} d_1 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ e_1 & d_2 & 0 & & & & \cdot \\ f_1 & e_2 & d_3 & 0 & & & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & 0 & f_{n-2} & e_{n-1} & d_n \end{pmatrix}.$$

   b. Determine the precise number of floating point operations in your algorithm. Include all additions, subtractions, multiplications, divisions, and evaluations of square roots.

4. a. State the Gerschgorin Circle Theorem.
   b. Use the Gerschgorin Circle Theorem to prove that each strictly diagonally dominant real $n \times n$ matrix is nonsingular.

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Probability

1. From an urn containing n_R red balls, n_B blue balls, and n_W white balls three balls are drawn at random successively without replacement. Calculate the following probabilities.
   a. All three balls are red.
   b. At least one ball is red.
   c. One ball is red, one ball is blue, and one ball is white.
   c. Repeat the problem for drawing the balls randomly with replacement.

2. Let X and Y be independent random variables having Poisson densities with parameters λ_1 and λ_2 respectively. The density of X is

$$f(x) = \frac{\lambda^x e^{-\lambda}}{x!} \quad x = 0, 1, \ldots$$

   and the same for Y.
   a. Show that the moment generating function (MGF) of X is

$$M_X(t) = e^{\lambda(e^t - 1)}.$$

   Hint: Recall the Taylor series expansion of the exponential function,

$$e^\lambda = \sum_{x=0}^{\infty} \frac{\lambda^x}{x!}$$

   b. Let $Z = X + Y$. Find the density of Z using MGFs.
   c. Find $P(X = x \mid X + Y = z)$ for $x = 0, 1, \ldots, z$.
   d. Compute $E[X \mid X + Y = z]$ where z is a nonnegative integer.

Probability—continued

3. Suppose a factory has two machines, A and B, that make 60% and 40% of the total production, respectively. Of their output machine A produces 3% and machine B produces 5% defective items.
   a. Find the probability that the factory produces a defective item.
   b. Find the probability that a given defective item was produced by machine B.
   c. Find the probability that a given defective item was produced by machine A.

4. Let X be a random variable with probability density function

$$f(x) = \begin{cases} c(1-x^2) & -1 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

   a. Determine the value of c that makes $f(x)$ a density function.
   b. What is the cumulative distribution function of X?
   c. Compute the mode of the density function $f(x)$ and show that it is also the median.